

## RECENT PROGRESS IN LATTICE QCD

STEPHAN DÜRR

*Bergische Universität Wuppertal, Gaußstraße 20, 42119 Wuppertal, Germany  
 Jülich Supercomputing Center, Forschungszentrum Jülich, 52425 Jülich, Germany  
 E-mail: durr(AT)itp.unibe.ch*

Recent progress in Lattice QCD is highlighted. After a brief introduction to the methodology of lattice computations the presentation focuses on three main topics: Hadron Spectroscopy, Hadron Structure and Lattice Flavor Physics. In each case a summary of recent computations of selected quantities is provided.

## 1 Introduction

Lattice QCD has come of age. What is meant with this statement is that 40 years after its inception [1] this framework is now able to deliver clear-cut predictions for a number of phenomenologically relevant quantities.

From an experimentalist's viewpoint what matters is the precision/accuracy of a computation; to be relevant it must match the experiment's precision (which often is a challenge). From a theoretician's viewpoint the key issue is that all quantities are well defined and that the total uncertainty of the final result (usually split into statistical and systematic contributions) can be reliably assessed. In that respect the last decade has brought a major improvement: The quenched approximation (in which the quark loops that come from the functional determinant are omitted) is gone. While often found to have a small impact in practice (a posteriori, i.e. by comparing to the full/unquenched result), this approximation was a nuisance, because its impact was extremely hard to quantify. Today's lattice computations set new standards, since the collaborations spend an enormous amount of effort to control and quantify *all* sources of systematic uncertainty.

There is a number of physics questions in which Lattice QCD plays a key role. One example is "Do we understand the weight of this world" (the visible matter in the Universe) ? With the Higgs particle established at the  $4\sigma$  level (as reviewed at this conference [2,3]) one might think that this is the cause. Looking at actual numbers we realize that the Higgs mechanism is almost irrelevant for the mass of the proton; the average of the up and down quark masses is  $m_{ud}^{\text{phys}} \equiv (m_u^{\text{phys}} + m_d^{\text{phys}})/2 \simeq 3.5 \text{ MeV}$  in  $(\overline{\text{MS}}, 2 \text{ GeV})$  conventions, while the proton mass is  $M_p \simeq 940 \text{ MeV}$ . At this point we stand accused of comparing apples to pears; the former mass is a scheme and scale dependent quantity, while the latter one is a truly physical quantity. A more meaningful attempt would relate the proton mass to the mass that the nucleon would have in the 2-flavor chiral limit, that is to  $M_N \simeq 880 \text{ MeV}$  with  $m_{ud} \rightarrow 0$  and all other quark masses held fixed (cf. Sec. 4). Still, the message is the same: The weight of the world which we experience is essentially due to the relativistic dynamics that is involved in the process by which the strong force binds quarks and gluons into protons (and other color neutral objects).

To date the lattice is the only known theoretical tool which can “solve QCD” in practical terms and with fully controlled systematics. This means that it is capable of establishing the link between fundamental parameters of the Standard Model (e.g. quark masses or CKM matrix elements) and the quantities measured in experiment (e.g. hadron masses or leptonic and semi-leptonic widths or branching fractions). In the following the goal will be to explain the theoretical underpinning of these computations, and to highlight some of the latest results.

The remainder of this review is organized as follows. Section 2 gives an outline of the field-theoretic setup of Lattice QCD computations. The next three sections contain the core of the presentation, an update on Hadron Spectroscopy (Sec. 3), Hadron Structure (Sec. 4) and Flavor Physics (Sec. 5, with a brief presentation of the recent FLAG activities). To give the reader an idea of what a tiny segment of lattice studies has been covered, we shall attempt to list some of the topics omitted in Section 6 before presenting a summary in Section 7.

## 2 Lattice Field Theory

### 2.1 Regulating QCD in the UV and the IR

Quantum Chromodynamics (QCD) is a field theory. As such it requires, in an intermediate stage, regularization both in the ultraviolet (UV) and in the infrared (IR) to be mathematically well behaved.

The lattice approach (more precisely: the multitude of lattice approaches, see below) does this by replacing the spacetime manifold by a regularly spaced 4-dimensional grid. Typically, a hypercubic set of grid points  $x \in (n_1, n_2, n_3, n_4)a$  is used, with  $n_i$  ( $i = 1, 2, 3$ ) running from 1 to  $N_s$  and  $n_4$  running from 1 to  $N_t$  and  $a$  the lattice spacing. The temporal direction is considered the fourth direction, since we work in Euclidean space, where all directions have the same sign in the metric. The lattice spacing  $a$  is usually chosen isotropically, i.e. the same (with  $c = 1$ ) in Euclidean space and time directions (though other options are possible).

The matter and gauge fields live in these grid points and in the links that connect neighboring grid points, respectively. Either type of fields may have internal degrees of freedom (as indicated by spinor and/or color indices). For convenience one usually introduces periodic or antiperiodic boundary conditions. Hence, starting in  $(aN_s, y, z, t)$  and hopping one unit in the 1-direction one ends up in  $(a, y, z, t)$  and similarly for the other directions, except for the 4-direction, where one picks up a sign if one is a fermion. With this setup the total number of degrees of freedom is finite, while the degrees of freedom themselves (the quark and gluon fields) are continuous (if they are discrete, too, one has a spin system).

By introducing the rule that each configuration of quark and gluon fields shall be attributed the Boltzmann weight  $\exp(-S_{\text{QCD}})$ , where the action [4]

$$S_{\text{QCD}} = \frac{1}{2g^2} \text{Tr}(F_{\mu\nu} F_{\mu\nu}) + \sum_{i=1}^{N_f} \bar{q}^{(i)} (\not{D} + m^{(i)}) q^{(i)} + i\theta \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr}(F_{\mu\nu} F_{\rho\sigma}) \quad (1)$$

is the Euclidean counterpart of the Minkowskian Lagrangian, one defines the partition function of QCD, provided there is an unambiguous measure for the variables

in each grid point or link [which is true for compact gauge groups like  $SU(N_c)$ ]. This weight depends on the discretization or action chosen, i.e. on the details of how one expresses the quantities in (1) in terms of the quark fields  $q^{(i)}(x)$  and gauge links  $U_\mu(x)$ , where  $x$  now denotes a 4-dimensional coordinate [which is implicit in (1)]. For instance, the Dirac operator  $\mathcal{D}$  may be anti-hermitian (as one would expect from a continuum viewpoint) in one discretization, but not even a normal operator in another one. Phenomenologically, the parameter  $\theta$  is known to be extremely close to 0 (or  $\pi$ ), this is why the last term in (1) is usually omitted.

To cut a long story short, the discretization of a field theory is an intermediate step which renders the path-integral (or partition function) finite. This happens right from the outset, in sharp contrast to perturbative approaches where one tames, at a much later stage, integrations which otherwise would produce infinities. The lattice provides a UV cut-off (through  $a > 0$ ) and an IR-cut-off (through  $V = L^3 T < \infty$ , where  $L = N_s a$  is the box-length and  $T = N_t a$  the time-extent) that respects the gauge invariance of (1). One then computes inverse correlation lengths in lattice units, e.g.  $a/\xi_\pi = aM_\pi$  and  $a/\xi_\Omega = aM_\Omega$ . The rule is that one must form ratios of such quantities before one is allowed to take the continuum limit. Hence, while  $\xi_\pi/a$  would diverge, the ratio  $\xi_\pi/\xi_\Omega$  stays finite under  $a \rightarrow 0$ . The crucial point is that the result is independent of the details of the lattice action (1), provided some guidelines were observed. In other words, the various lattice regularizations provide different ways of defining QCD, but they share a universal continuum limit. Any valid discretization of (1) is thus part of the *definition* of QCD !

## 2.2 Scale hierarchies in Lattice QCD

With this bit of lattice ideology in place, it is time to reflect on something relevant in every days use of Lattice QCD, namely the scale hierarchies involved. Evidently, the shortest length-scale at hand is the lattice spacing  $a$ , the largest one is the box size  $L$ . The correlation lengths of the quarks that occur in (1) and of the hadrons that emerge as bound states should somehow lie in between these extremes.

It turns out that for the shortest scales it is the (heaviest) quark mass that matters, while for the largest scales it is the mass of the (lightest) asymptotic state that is relevant. In other words, for the quarks we need to pay attention that they do not “fall through the grid” (i.e. we request  $am_q \ll 1$ ), while for the pions we need to make sure that they are not squeezed too much (i.e. we request  $M_\pi L \gg 1$ ). This may be a bit of a challenge, because maintaining both conditions simultaneously requires having a large number of grid points in each direction.

At the time of writing the largest number of grid points that can be sustained in current state-of-the-art lattice QCD simulations is of the order  $128^4$ . Even with this somewhat optimistic assumption on  $L/a$  the goal  $M_\pi L > 4$  would translate, at the physical pion mass, into  $L = 4.1/(135 \text{ MeV}) = 6 \text{ fm}$  and hence into  $a = 0.047 \text{ fm}$ . With  $m_c \simeq 1.1 \text{ GeV}$  the product is then  $am_c = 0.26$ . This number is small enough, but in order to take a continuum limit we need to have several lattice spacings. We can only afford to go to larger  $a$ ; upon doubling the lattice spacing we end up with  $am_c = 0.52$  which (with the actions currently in use) is barely acceptable.

Hence simulating almost physical light quarks and, at the same time, charm

quarks without terrific cut-off effects poses enormous computational requirements. Usually one chooses for the up and down quarks a common mass  $m_{ud}$ , and either this mass is kept in the vicinity of its physical (isospin averaged) value or  $m_c$  is kept near its physical mass. Often some extrapolation in  $m_{ud}$  and/or  $m_c$  is performed, while  $m_s$  is usually simulated next to its physical mass value. Simulating bottom quarks in the vicinity of their physical mass is out of question (except for special setups); they are usually treated in entirely different frameworks. And the top is omitted from Lattice QCD calculations, since it is too short lived to hadronize.

### 2.3 How to remove systematic effects

As emphasized in subsec. 2.1, the lattice (or something equivalent) is a necessary intermediate step in the definition of QCD. In consequence, a computer code which yields a stochastic estimate [5] of the path integral (1) for a given lattice spacing  $a$  and box size  $L$ , at a given set of quark masses  $m_q = (m_{ud}, m_s)$  [and possibly  $m_c$ ], does not complete the job. A good lattice calculation combines the information from several simulations to remove the remnants of the lattice formulation. Thus, when reading a lattice paper one should have the following questions in mind:

- (1) Has the continuum limit ( $a \rightarrow 0$ ) been taken ?
- (2) Are the finite-volume effects (from  $L < \infty$ ) under control ?
- (3) Are the simulations performed anywhere close to  $M_\pi = 135$  MeV ?
- (4) Advanced: are theoretical uncertainties properly assessed/propagated ?
- (5) Experts: algorithm details, treatment of isospin breakings, resonances, ...

Unfortunately, all the interesting directions tend to be expensive in terms of computer time. The computational requirements tend to increase roughly like

$$\text{CPU} \propto 1/a^{4-6}, \quad \text{CPU} \propto L^5, \quad \text{CPU} \propto 1/m_q^{1-2} \quad (2)$$

and the meaning is that in each case the other parameters are held fixed. In reality, when pushing  $m_{ud}$  down towards  $m_{ud}^{\text{phys}} \simeq 3.5$  MeV also the box length  $L$  needs to be increased to maintain the bound  $M_\pi L > 4$ . In addition, there are algorithmic issues which lead to a proliferation of noise near the chiral limit [6]; this makes simulations close to the physical mass point even more demanding.

The quark fields in (1) may be integrated out, and this leads to a contribution  $\sum \log(\det(\not{D} + m^{(i)}))$  to the effective action, where the sum runs over the flavors,  $i = 1, \dots, N_f$ . Today's lattice terminology attributes the label  $N_f = 2$  to simulations with a common mass  $m_{ud}$  of the “sea” quarks (i.e. those which come from the functional determinant). Studies which include a dynamical strange and possibly a dynamical charm quark are referred to as  $N_f = 2 + 1$  and  $N_f = 2 + 1 + 1$  simulations, respectively, to indicate that these quarks have separate masses.

The lattice spacing and the quark masses cannot be dialed “a priori” because of renormalization effects. The simulations are governed by the bare gauge coupling  $\beta = 2N_c/g^2$  and several (in the cases mentioned above: 2 or 2 + 1 or 2 + 1 + 1) bare mass parameters; the lattice spacing  $a$  and the quark masses  $m_q$  are emergent quantities to be determined “a posteriori”.

### 3 Hadron Spectroscopy

#### 3.1 Quenched versus unquenched QCD

Spectroscopy of stable states with a conserved quantum number (e.g. isospin) is about the easiest thing to do in Lattice QCD. One considers two-point functions

$$C(t) = \langle O(t)O^\dagger(0) \rangle = \frac{1}{Z} \int DU \underbrace{\hat{O}(t)\hat{O}^\dagger(0)}_{\text{"valence"}} \prod_{i=1}^{N_f} \underbrace{\det(\not{D}[U] + m^{(i)})}_{\text{"sea"}} e^{-S[U]} \quad (3)$$

where  $O(\cdot)$  is designed to absorb these quantum numbers, and at least one of the two involves some projection to a definite spatial momentum (e.g.  $\mathbf{p} = 0$ ). From the asymptotic behavior  $C(t) \propto e^{-E(\mathbf{p})t}$  one extracts the mass of the particle. In the past the functional determinant was omitted ( $N_f = 0$ ), while  $O$  would still contain so-called valence quarks; this is the quenched approximation. If a quark mass in the determinant is different from the mass of the same flavor in the interpolating fields  $O(\cdot)$ , i.e.  $m_{ud}^{\text{sea}} \neq m_{ud}^{\text{val}}$ , one says that the respective flavor is partially quenched.

#### 3.2 Landscape of current $N_f = 2 + 1$ simulations

As discussed before, generating ensembles with light  $m_{ud}$  tends to be expensive. Fig. 1 depicts the sea pion masses and box sizes at which various collaborations managed to simulate. A significant number of ensembles is needed to extrapolate to  $(m_{ud}^{\text{phys}}, m_s^{\text{phys}})$  and to remove the cut-off and finite-size effects. To reach the physical point most collaborations fix  $m_s \simeq m_s^{\text{phys}}$  and reduce  $m_{ud}$  as much as possible (as of this writing only two collaborations can bracket  $m_{ud}^{\text{phys}}$  by the  $m_{ud}$  in the sea [8,9]). By contrast, QCDSF pursues an interesting alternative. They start with a common sea quark mass near  $(2m_{ud}^{\text{phys}} + m_s^{\text{phys}})/3$  and split the masses symmetrically such that this weighted sum stays invariant [10].

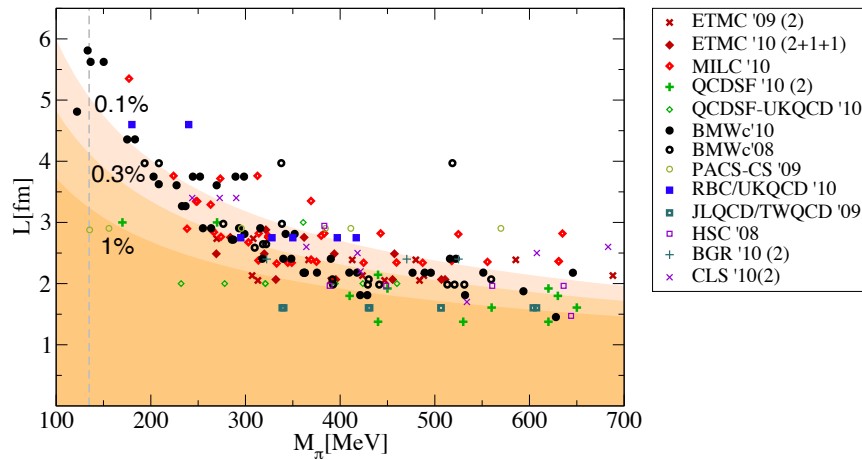


Figure 1. Landscape of simulated box sizes and pion masses in recent studies of full QCD. The amber shaded areas indicate the expected magnitude of relative finite size effects. Figure from [7].

### 3.3 Reality check: spectra of stable hadrons

We tend to postulate that the QCD Lagrangian (1) is the complete theory of strong interactions, valid both in the short-distance regime (where asymptotic freedom prevails) and in the long-distance regime (where confinement dominates).

This is a highly non-trivial statement, and it was a historical achievement of the CP-PACS collaboration to show that quenched QCD (as an alternative candidate) does not pass the test [11] (left panel in Fig. 2). The unquenched version fares much better [12] ( $N_f = 2 + 1$ , right panel in Fig. 2); of course, this does not amount to a proof. In the latter case three quantities are plotted with open circles, as they have been used to adjust  $m_{ud}$ ,  $m_s$  and to determine the individual  $a$  (cf. Sec. 2).

### 3.4 Resonances and excited baryon spectra

Those who wish to tackle harder problems may study how resonances interact with stable particles (note that the lattice results in Fig. 2 concern only their masses; the gray bands indicate experimental widths). A prominent example is  $g_{\rho\pi\pi}$ , the coupling of the rho to two pions, on which there are nice results; see [13] for a review. In the same category is the study of excited states with more mundane quantum numbers (e.g. those of the proton), see e.g. [14] for recent progress.

### 3.5 Mixing of $\eta - \eta'$

One of the long standing problems of QCD is to convincingly show that the mass splitting between the  $\eta$  and  $\eta'$  is due to the global axial anomaly (as argued by Witten and Veneziano long ago). This is now achieved on a quantitative level (i.e. with mixing) in two recent papers, one by RBC/UKQCD [15], one by ETMC [16].

### 3.6 Progress on glueballs

Glueballs are hard in pure Yang-Mills theory (because they are noisy, and sometimes a vacuum contribution needs to be subtracted), and they are extremely hard in the presence of dynamical flavors (since they mix with other flavor singlet states which do have valence quarks). For a recent paper on the subject see [17].

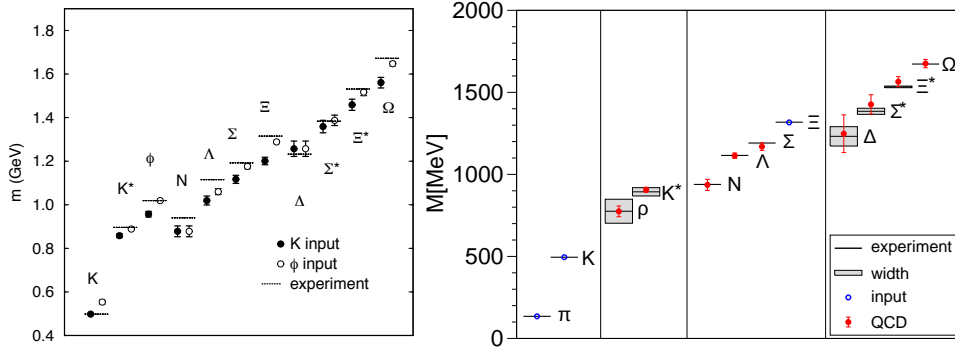


Figure 2. Light pseudoscalar and vector meson spectra after  $a \rightarrow 0$ ,  $L \rightarrow \infty$  in quenched QCD (left,  $N_f = 0$ , published in 2000 [11]) and in full QCD (right,  $N_f = 2 + 1$ , published in 2008 [12]).

## 4 Hadron Structure

### 4.1 Nucleon sigma terms and dark matter searches

Dark matter experiments try to probe the interaction of unknown objects (WIMPs) with nuclei. This would happen via coupling to virtual light and strange quark loops in the nucleon, and the lattice is thus called to calculate the quantities

$$\sigma_{\pi N} = \sigma_{ud} = m_{ud} \langle N | u\bar{u} + d\bar{d} | N \rangle, \quad \sigma_s = m_s \langle N | s\bar{s} | N \rangle \quad (4)$$

from first principles. This can be done either from the slope of  $M_N$  versus  $M_\pi^2$  and  $2M_K^2 - M_\pi^2$  (see Fig. 3) or from a direct determination of the matrix elements. A straight average of all central values and total errors in Tab. 1 would suggest that

$$\sigma_{\pi N} = \sigma_{ud} = 47(9) \text{ MeV}, \quad \sigma_s = 48(25) \text{ MeV} \quad (5)$$

is a conservative estimate (in some cases  $\sigma_s$  is obtained from  $y_N = 2m_l/m_s \cdot \sigma_s/\sigma_{ud}$  or  $f_{T_s} = m_s \langle N | s\bar{s} | N \rangle / M_N$ ), see also [29]. With  $\sigma_{\pi N} \simeq M_\pi^2 \partial(M_N)/\partial(M_\pi^2)$  it follows that the nucleon mass in the 2-flavor chiral limit is  $M_N(m_{ud}=0) \simeq 880(20) \text{ MeV}$ .

### 4.2 Nuclear structure and recent news on $g_A$

The form factors of the nucleon with external  $S, P, V, A$  currents and the respective radii are known to be hard on the lattice. Even the values at  $\mathbf{p}^2=0$ , in case of the axial current known as  $g_A$ , turn out to be unexpectedly hard [30,31] (see also [32]).

50(9)(3)	33(16)(5)	Young Thomas 09 [18]
75(15)	—	PACS-CS 09 [19]
—	59(6)(8)	Toussaint Freeman 09 [20]
39(4)( $^{+18}_{-7}$ )	34(14)( $^{+28}_{-24}$ )	BMW-c 10 [21]
31(3)(4)	71(34)(59)	QCDSF/UKQCD 11 [22]
—	40(7)(5)	MILC 12 [23]
45(6)	21(6)	Shanahan et al 12 [24]
—	8(14)(15)	JLQCD 12 [25]
43(1)(6)	126(24)(54)	Ren et al 12 [26]
—	43(10)	Engelhard 12 [27]
—	49(10)(15)	Junnarkar Walker-Loud 13 [28]

Table 1. Summary of recent  $N_f = 2 + 1$  lattice determinations of  $\sigma_{\pi N}$  and/or  $\sigma_s$ .

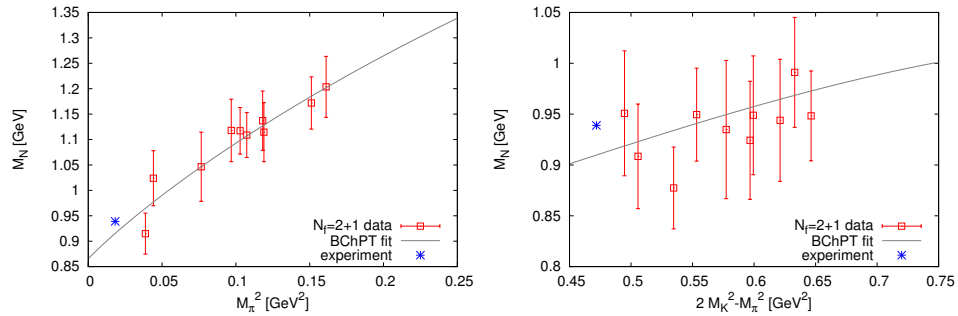


Figure 3. The nucleon mass as a function of the average light quark mass  $m_{ud} \propto M_\pi^2$  at fixed physical  $m_s \propto 2M_K^2 - M_\pi^2$  (left) and vice versa (right) in full QCD simulations [21].

#### 4.3 Scattering of $\pi\pi$ , $\pi K$ , $KK$ , $\pi N$ , $NN$ and more

Elastic scattering parameters can be extracted from the  $L$ -dependence of accurately measured energies of two-body states in a finite volume [33], cf. Fig. 4 left. Recently a new method has been proposed to extract such information from two-body potentials, and it is claimed that such results are less sensitive to excited states contaminations [34]. The right panel shows the central  $NN$  potential (spin-singlet channel) determined with this method; both the repulsion at short distance and the attraction at intermediate distance seem to grow as the pion mass decreases.

#### 4.4 From quarks to nuclei

Evidently, there is a long way to go until all nuclear physics can be derived from Lattice QCD; see Fig. 5 and [35,36]. A method to deal with the vastly growing number of contractions is proposed in [37]. The subject is reviewed in [38].

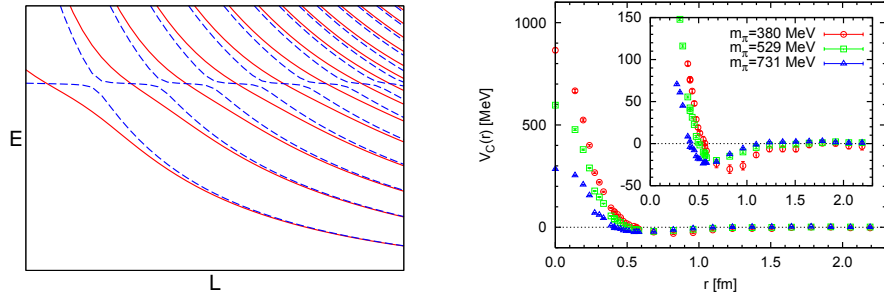


Figure 4. Principle of extracting scattering lengths and phase shifts from avoided level crossings in a finite volume (left, “level method” [33], figure from [7]) and result for central nucleon-nucleon potential in  $^1S_0$  state at various pion masses (right, “potential method”, figure from [34]).

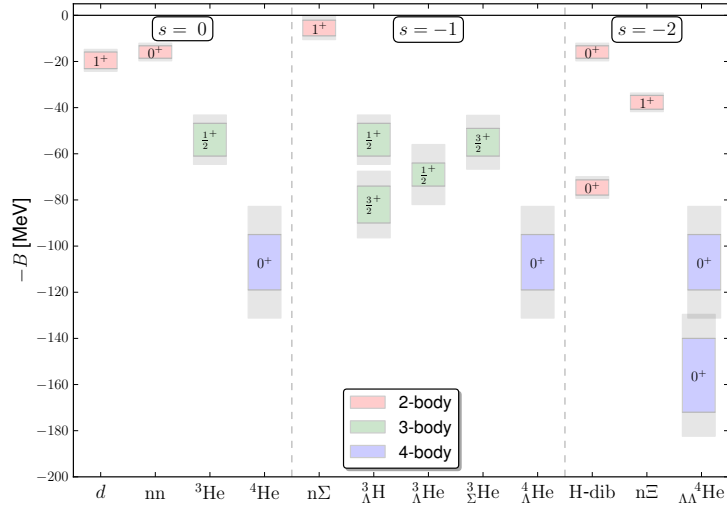


Figure 5. A compilation of the nuclear energy levels, with spin and parity, as determined in [35].



## 5 Flavor Physics and FLAG

### 5.1 Light quark masses and $\alpha_{\text{strong}}$

The quark masses and  $\alpha_{\text{strong}}$  are the parameters of the Standard Model that Lattice QCD can determine without further theoretical input. The starting point is a spectroscopy calculation like the one shown in the right panel of Fig. 2. With this one has the quark masses in hand, albeit in an awkward scheme which is specific to the lattice action used in the course of the simulation. Before taking the continuum limit a conversion to a continuum scheme (e.g.  $\overline{\text{MS}}$  or RI/MOM or SF) must be performed. A few recent computations are listed in Tab. 2.

Similarly, by observing how the lattice spacing  $a$  varies as one shifts the coupling  $\beta = 2N_c/g^2$  or by matching small Wilson loops to lattice perturbation theory one can determine the QCD  $\beta$ -function and the integration constant  $\Lambda_{\text{QCD}}$  (the case  $N_f = 2$  has been presented at this conference [44]) or  $\alpha_{\text{strong}}(\mu)$  at a scale  $\mu \sim a^{-1}$ . The PDG average of  $\alpha_{\text{strong}}(M_Z)$  is dominated by the HPQCD results [45,40].

### 5.2 Light decay constants and form factors

The leptonic decay of a pseudoscalar meson ( $P = \pi, K, \dots$ ) is mimicked on the lattice by the coupling to an external axial current and captured in the decay constant  $f_P$ . Experiment determines the width  $\Gamma \propto |f_\pi V_{ud}|^2$  or  $\Gamma \propto |f_K V_{us}|^2$ , and by dividing out  $f_P$  one gets access to the CKM matrix element  $V_{ud}$  or  $V_{us}$ , respectively.

Similarly, the lattice can determine the  $K \rightarrow \pi$  transition form factor (with a vector current in between)  $f_{+,0}^{K \rightarrow \pi}(q^2)$ , and by combining it with the width of the semileptonic  $K \rightarrow \pi$  decay one has another option to access  $V_{us}$ .

In either case, when computing  $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2$  (the last contribution is tiny) one finds that the first-row CKM unitarity relation is well satisfied.

### 5.3 Kaon mixing in and beyond the standard model

The  $K^0$ - $\bar{K}^0$  oscillation is the main source of indirect CP violation. In the standard model (SM) it is dominated by two box diagrams which, via an effective field theory approach, may be shrunk into a  $\Delta S = 2$  operator, such that one ends up with  $\langle \bar{K}^0 | O_{VV+AA} | K^0 \rangle$ . The lattice can calculate this matrix element (which enters the unitarity triangle analysis) or  $B_K$ , see e.g. [46,47,42,43], as well as siblings which are relevant to beyond standard model (BSM) theories [48,49].

Recently RBC/UKQCD managed to calculate the  $K \rightarrow (2\pi)_{I=2}$  amplitude (both Re and Im of this  $\Delta I = 3/2$  process) [50]. Still, the  $\Delta I = 1/2$  counterpart  $K \rightarrow (2\pi)_{I=0}$  (relevant to the  $\Delta I = 1/2$  rule and  $\epsilon'/\epsilon$ ) is significantly harder.

88(0)(5)	3.2(0)(2)	1.9(0)(2)	4.6(0)(3)	MILC [39]
92.2(1.3)	—	2.01(10)	4.77(15)	HPQCD [40]
95.5(1.1)(1.5)	3.469(47)(48)	2.15(03)(10)	4.79(07)(12)	BMW-c [41]
94.2(1.4)(5.7)	3.31(07)(26)	1.90(08)(23)	4.73(09)(36)	LVdW [42]
92.3(1.9)(1.3)	3.37(09)(07)	—	—	RBC/UKQCD [43]

Table 2. Selection of recent computations of  $m_s, m_{ud}, m_u, m_d$  [MeV] in the  $(\overline{\text{MS}}, 2 \text{ GeV})$  scheme.

#### 5.4 FLAG compilation and lattice averages

The quantities discussed in the previous subsections have been evaluated by many lattice groups. In such a situation it is useful to have reviews which provide averages, like the one by FLAG [51] (with a focus on low-energy physics) or [52] (with an eye on CKM physics). In either work the user is urged to cite the original literature. Recently these two teams merged into FLAG-II with the goal to provide a regularly updated compilation of a larger number of observables, with trustworthy assessment of the overall systematic uncertainty: <http://itpwiki.unibe.ch/flag>.

#### 5.5 Charm physics on the lattice

Treating the charm quark on the lattice just became doable (cf. Sec. 2). Recent years brought several precise determinations of (leptonic) decay constants and (semi-leptonic) transition form factors of  $D$  and  $D_s$  mesons [essentially the same story as was told for pions and kaons before]; see Fig. 6 and Tab. 3 for a tiny selection.

#### 5.6 Bottom physics on the lattice

Due to its large mass the bottom quark tends to be treated with specialized methods (cf. Sec. 2). Mass splittings in the  $\Upsilon$  system and again decay constants and form factors are of interest; see Tab. 4 for a spotlight and [61,62] for recent reviews.

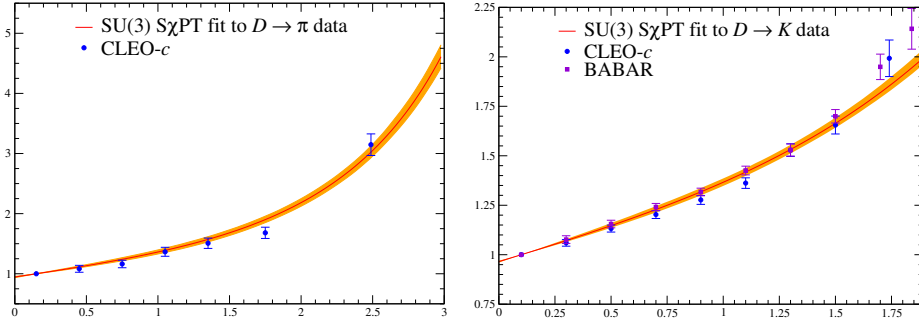


Figure 6. Shape of the form factors  $f_+^{D \rightarrow \pi}(q^2)$  (normalized at  $q^2 = 0.15 \text{ GeV}^2$ , left) and  $f_+^{D \rightarrow K}(q^2)$  (normalized at  $q^2 = 0.10 \text{ GeV}^2$ , right) compared to experiment. Figure from [53].

226(6)(1)(5)	257(2)(1)(5)	1.14(3)(0)(0)	PACS-CS [54]
212(8)	248(6)	1.17(5)	ETMC [55]
208.3(1.0)(3.3)	246.0(0.7)(3.5)	1.187(04)(12)	HPQCD [56]
209.2(3.0)(3.6)	246.4(0.5)(3.6)	1.175(16)(11)	FNAL/MILC [57]

Table 3. Selection of recent computations of  $f_D$  [MeV],  $f_{D_s}$  [MeV] and  $f_{D_s}/f_D$ .

195(12)	232(10)	1.19(5)	ETMC [55]
—	225(04)	—	HPQCD [58]
196.9(9.1)	242.0(10.0)	1.229(26)	FNAL/MILC [59]
191(9)	228(10)	1.188(18)	HPQCD [60]

Table 4. Selection of recent computations of  $f_B$  [MeV],  $f_{B_s}$  [MeV] and  $f_{B_s}/f_B$ .

## 6 Other Topics

### 6.1 Large $N_c$ , larger $N_f$ , different representations

Having a computer code which generates full QCD ensembles, it is straightforward to change the number of colors,  $N_c$ , the number of dynamical fermions,  $N_f$ , or even their color representation. Most practitioners are interested in the regime where such theories are slowly walking, i.e. near conformality, see [63,64] for reviews.

### 6.2 QCD thermodynamics at $\mu = 0$ and $\mu > 0$

At vanishing baryon number density ( $\mu=0$ ) and physical quark masses QCD shows a crossover [65,66]. Recently, a consensus was reached on the pseudocritical temperatures  $T_c$  for a set of common observables (subtracted chiral condensate, Polyakov loop susceptibility) at physical quark masses and in the continuum [66,67,68].

QCD thermodynamics at non-zero chemical potential ( $\mu > 0$ ) is in a much less satisfactory state. The main reason is that there is a genuine sign problem in the regime  $\mu \gg T, M_\pi$  (without any solution in sight). There are established techniques to investigate the regime  $\mu \ll T, M_\pi$  (which is good enough for cosmology, since the strong transition was at  $\mu \simeq 0$ ). To date it is not clear whether there is a line of first-order transitions separating the confined (hadronic) phase from the deconfined (plasma) phase. If it exists, it would terminate in a second-order endpoint [69].

### 6.3 Hadronic contributions to $g-2$ of the muon

Hadronic contributions to vacuum polarization provide one of the major sources of systematic uncertainty in the computation of the anomalous magnetic moment of the muon  $a_\mu \equiv (g_\mu - 2)/2$ . The lattice can help by providing the (Fourier transformed) 2-point function of the electromagnetic vector current in such a hadronic environment. Recent papers include [70,71,72]; see also [73].

### 6.4 QCD with isospin splitting and/or electromagnetism

In current state-of-the art studies of lattice QCD photon loops are excluded from the simulations, and strong isospin breaking (due to  $m_u \neq m_d$ ) is ignored (since it is of the same order of magnitude). To compensate, input quantities (e.g.  $M_\pi, M_K, M_\Xi$  in Fig. 2 right) are usually adjusted for these effects, see [51] for details.

Recently, there has been significant progress in exploratory studies which include quenched photons (i.e. interacting with the valence but not with the sea quarks). The key issue is that the finite-volume effects are no longer exponentially small for large  $M_\pi L$ , but only polynomially suppressed (see e.g. [74]). Similarly, there has been tremendous progress in splitting  $m_u \neq m_d$  through reweighting techniques (see [75], though in this work a considerable fraction of the reweighting power goes into shifting the average light quark mass downwards).

The ultimate dream of today's lattice physicists is to perform  $N_f = 1+1+1+1$  simulations, such that the physical values of  $m_u, m_d, m_s, m_c$  are bracketed by those in the ensembles, and the lattices would include the photon as a dynamical degree of freedom and cover, furthermore, several lattice spacings  $a$  and box sizes  $L$  such that an extrapolation  $a \rightarrow 0$  and  $L \rightarrow \infty$  can be performed.

### 6.5 Towards exaflop machines

Among numerically oriented scientists of all disciplines the lattice people are known for their hunger for computer time. Current flagship machines tend to provide  $O(10)$  petaflops of peak performance [about  $10^{16}$  multiply-and-add cycles per second], with a power consumption of the order of 4 MW. Reaching the exaflop era ( $10^{18}$  flops) requires new thoughts on how to make these machines even more energy efficient, since scaling the electricity bill up by two orders of magnitude is not an option.

The linpack tests that are used to rank such machines on the “top 500” list use about 80% of the available cycles. For typical lattice codes this sustained performance ratio ranges between 20% and 50% (which is significantly higher than what is reached in most other disciplines). Still, with every new generation of supercomputers the amount of parallelism grows, and it is an ever increasing challenge to adjust the code such that these high sustained performance figures would persist.

### 6.6 More theoretical issues

Let me just mention some of the topics on which either tremendous progress has been achieved or which mark open questions: improved actions and matching with perturbation theory, chiral symmetry in vector-like gauge theories, chiral gauge theories and CP violation, chiral symmetry and chemical potential, the generic sign problem at non-zero chemical potential, how one would formulate supersymmetry on the lattice, further particulars of the staggered fourth-root procedure, minimally doubled fermions and staggered fermions with non-standard mass terms, the issue of large autocorrelation times near the chiral and/or continuum limit.

## 7 Summary

Instead of giving a long summary, I try to formulate seven short messages:

1. The discretization of the QCD Lagrangian (1) [or something equivalent] is a necessary intermediate step in the *definition* of QCD.
2. Spectroscopy of stable hadrons with  $N_f = 2$  or  $N_f = 2 + 1$  or  $N_f = 2 + 1 + 1$  active (dynamical) quarks is a mature field.
3. Spectroscopy of mixing or unstable [under strong interactions] states (often performed on the same gauge backgrounds) is developing fast.
4. Lattice QCD yields vital input in CKM analysis and BSM bounds, much of which is reviewed in the compilations by FLAG [51] and latticeaverages [52]. Use these resources, but please be sure to cite the original papers !
5. There is rapid progress on a number of nuclear issues, such as the nucleon sigma term and various scattering lengths and phase-shifts.
6. There is rapid progress on QCD thermodynamics. At  $\mu = 0$  a consensus on the relevant pseudocritical temperatures  $T_c$  has been reached, while at  $\mu \neq 0$  it is still not clear whether a second-order endpoint exists.
7. The lattice remains a relevant tool for conceptual work in field theory. For instance, it is still not clear whether supersymmetry is a well-defined field-theoretic concept (beyond perturbation theory).

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